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DEPARTMENT OF MATHEMATICS

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In 1772, Lagrange initiated the study of the permutations while studying the solutions of algebraic equations.

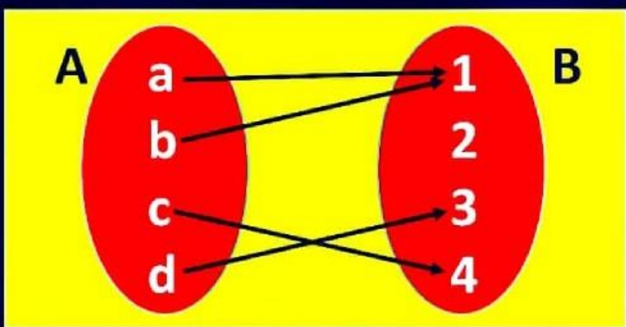
Galois(1812- 1832), first time use the word "Group" in a technical sense.

Basic Concepts

❖ Set: A collection of well-defined objects.

❖ Function : $\forall \ a \in A \ \exists \ a \ unique \ b \in \mathrm{B} \ni$

$$f(a) = b$$



❖ Binary Operation :

A way of computing from $S \times S \rightarrow S$

Group:

Let G be a non empty set and $*$ *be a binary operation on G. Then*

G_1: $\forall a, b \in G, a * b \in G$

G_2: $\forall a, b, c \in G,$

$$(a * b) * c = a * (b * c)$$

G_3: $\forall a \in G, \exists e \in G \ni a * e = e * a = a$

G_4: $If\ a \in G, \exists some\ b \in G \ni$

$$a * b = b * a = e.$$

If G satisfies

G_1 & $G_{2,}$ *then G is called* *semi Group*.

G_1, G_2&$G_{3,}$ *then G is called* *monoid.*

$G_1, G_{2,} G_3$&$G_{4,}$ *then G is called* a Group.

❖ In addition, $\forall a, b \in G, a * b = b * a$, then the group $(G,*)$ is called an abelian group.

Subgroup

Let $(G,*)$ *be a group and H be a non empty subset of G. Then H is said to be a subgroup, if* (H,∗) itself is a group.

Example: Let H={1,-1} sub set of a group G=$\{1, -1, i, -i\}$ *and* "." binary operation on G. Then, (H,.) is a subgroup of group G.
Since H itself is a group w.r.t. binary operation "."

.	1	-1	*i*	*-i*
1	1	-1	*I*	*-i*
-1	-1	1	*-i*	*i*
i	*i*	*-i*	-1	1
-i	*-i*	*i*	1	-1

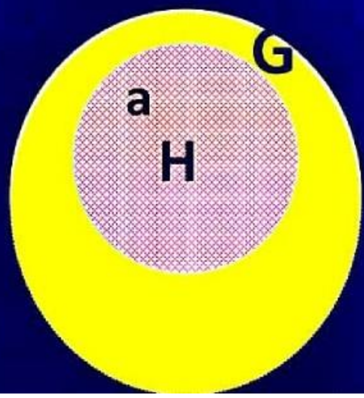
.	1	-1
1	1	-1
-1	-1	1

Cosets

Let $(H,.)$ *be a subgroup of a group* $(G,.)$ *and* $a \in G$. *Then*

$a.H = \{a.h | h \in H\}$ *is called a left Coset*

$H.a = \{h.a | h \in H\}$ *is called a right Coset of H containing a.*



Let $(H,+)$ *be a subgroup of a group* $(G,+)$ *and* $a \in G$. *Then, the left and right cosets are*

$$\boldsymbol{a + H = \{a + h | h \in H\}}$$

$$\boldsymbol{H + a = \{h + a | h \in H\}}$$

Here onwards,

1. G *stands for a group with the binary opeation* "."
2. H *stands for a subgroup of a group* $(G, .)$.
3. We write aH instead of $a.H$ and
4. We write ah instead of $a.h$

<u>Here are a few questions we try to address in this lecture:</u>

1. When is aH = bH or when is Ha = Hb?
2. Are all cosets subgroups? If not, which ones are?
3. $|aH| = |bH|$?
4. Can different cosets have elements in common?
5. Is $|aH| = |Ha|$?

Example 01:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}$$
$$H = \{0, \pm 2, \pm 4, \pm 6, \pm 8, \ldots\}.$$

$(H,+)$ is subgroup of $(\mathbb{Z},+)$.

Let $1 \in \mathbb{Z}$.

Then left & right cosets are

$$1 + H = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$
$$H + 1 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$$

Example: Let $G = \{1, -1, i, -i\}$ be a group and $H = \{1, -1\}$ be a subgroup of G. Then

Left cosets	Right cosets
$1H = \{1, -1\}$	$H1 = \{1, -1\}$
$-1H = \{-1, 1\}$	$H(-1) = \{-1, 1\}$
$iH = \{i, -i\}$	$Hi = \{i, -i\}$
$-iH = \{-i, i\}$	$H(-i) = \{-i, i\}$

Remark.

1. For an abelian group G,
   $aH = Ha$, i.e. right and left cosets of any subgroup coincide.
2. The subgroup H itself is always one of its left cosets and right cosets (since $eH = He = H$).

Theorem:01

Let H be a subgroup of a group G and $a \in G$. Then 1. $a \in aH$
2. $a \in Ha$.

Proof:

Since H is a sub group of group G, we have $e \in H$.

Then $a = ae \in aH$.

Similarly, we prove that $a \in Ha$.

Theorem:02

Let H be a subgroup of a group G and $a \in G$. *Then* $\mathbf{a \in H \Leftrightarrow aH = H.}$

Proof: *Suppose* $aH = H$.

By Theorem 01, $a \in aH$. But $aH = H$

Thus, we get $a \in H$.

Conversely, Suppose $a \in H$.

Claim: $aH = H\ (aH \subseteq H\ \&\ H \subseteq aH)$.

Let $ah \in aH$ *for some* $h \in H$.

Since H is a subgroup & a,h∈ H, *we get* $ah \in H$. Thus $aH \subseteq H$.

Let $h \in H$.

Since $a \in H$ *and* H *is a sub group,* we get that $a^{-1} \in H \ni aa^{-1} = e$.

$$\text{Now } h = eh = (aa^{-1})h$$
$$= a(a^{-1}h) \in aH.$$

Thus $H \subseteq aH$. *Hence* $H = aH$.

Theorem 04: Let H be a subgroup of a group G and $a, b \in G$. Then

1. $a \in bH \Leftrightarrow aH = bH$
2. $a \in Hb \Leftrightarrow Ha = Hb$.

Proof: Suppose $aH = bH$.

If $a \in aH$, then $a \in bH$

$(\because aH = bH)$.

Conversely, Suppose $a \in bH$.

$$\begin{aligned} Then\ b^{-1}a &\in b^{-1}bH \\ \Rightarrow b^{-1}a &\in eH \\ \Rightarrow b^{-1}a &\in H \\ \Rightarrow bH &= aH. \end{aligned}$$

Similarly, we can prove $a \in Hb \Leftrightarrow Ha = Hb$.

Example: Let $H = \{1, -1\}$ be a sub-group of a group $G = \{1, -1, i, -i\}$.
Then the left cosets are

$$1H = \{1, -1\}$$
$$-1H = \{-1, 1\}$$
$$iH = \{i, -i\}$$
$$-iH = \{-i, i\}$$

Here $1H = \{1, -1\} = -1H$

$iH = \{i, -i\} = -iH$

Theorem 05: *Let H be a subgroup of a group G and $a, b \in G$. Then any two left(right) cosets either identical or disjoint (or)*

If aH, bH be two left cosets of H in G, then either $aH \cap bH = \phi$ or $aH = bH$.

Proof: Let aH, bH be two left coests of H in G.

$Suppose\ aH \cap bH = \phi.$

Then nothing to prove.

Suppose aH $\cap\, bH \neq \phi.$

$$Then\ c \in aH \cap bH$$
$$\Rightarrow c \in aH\ \&\ c \in bH$$
$$\Rightarrow cH = aH\ \&\ cH = bH$$
$$(\because \text{ by Theorem } 04)$$
$$\Rightarrow aH = bH.$$

Similarly, we can prove either

$$Ha \cap Hb = \phi \text{ or } Ha = Hb.$$

Theorem 07: *Let H be a subgroup of a group G. Then there exists a bijection between the set of distinct left cosets of H in G and the set of distinct right cosets of H in G*

$Proof$: *Let*

$L_H =$ the set of all left cosets of H in G

$R_H =$ the set of all right cosets of H in G

Define f: $L_H \rightarrow R_H$ by $f(aH) = Ha^{-1}$, $\forall a \in G$.

$\underline{f \text{ is well} - \text{define}}$:

Suppose $aH = bH$

$\Rightarrow a^{-1}b \in H$

$\Rightarrow (a^{-1}b)^{-1} \in H$ ($\because$ H is a sub gp)

$\Rightarrow b^{-1}(a^{-1})^{-1} \in H$

$\Rightarrow Ha^{-1} = Hb^{-1}$

$\Rightarrow f(aH) = f(bH)$

$\underline{f \text{ is one} - \text{one}}$:

Suppose $Ha^{-1} = Hb^{-1}$

$\Rightarrow a^{-1}(b^{-1})^{-1} \in H$

$\Rightarrow a^{-1}b \in H$ ($\because$ H is a sub gp)

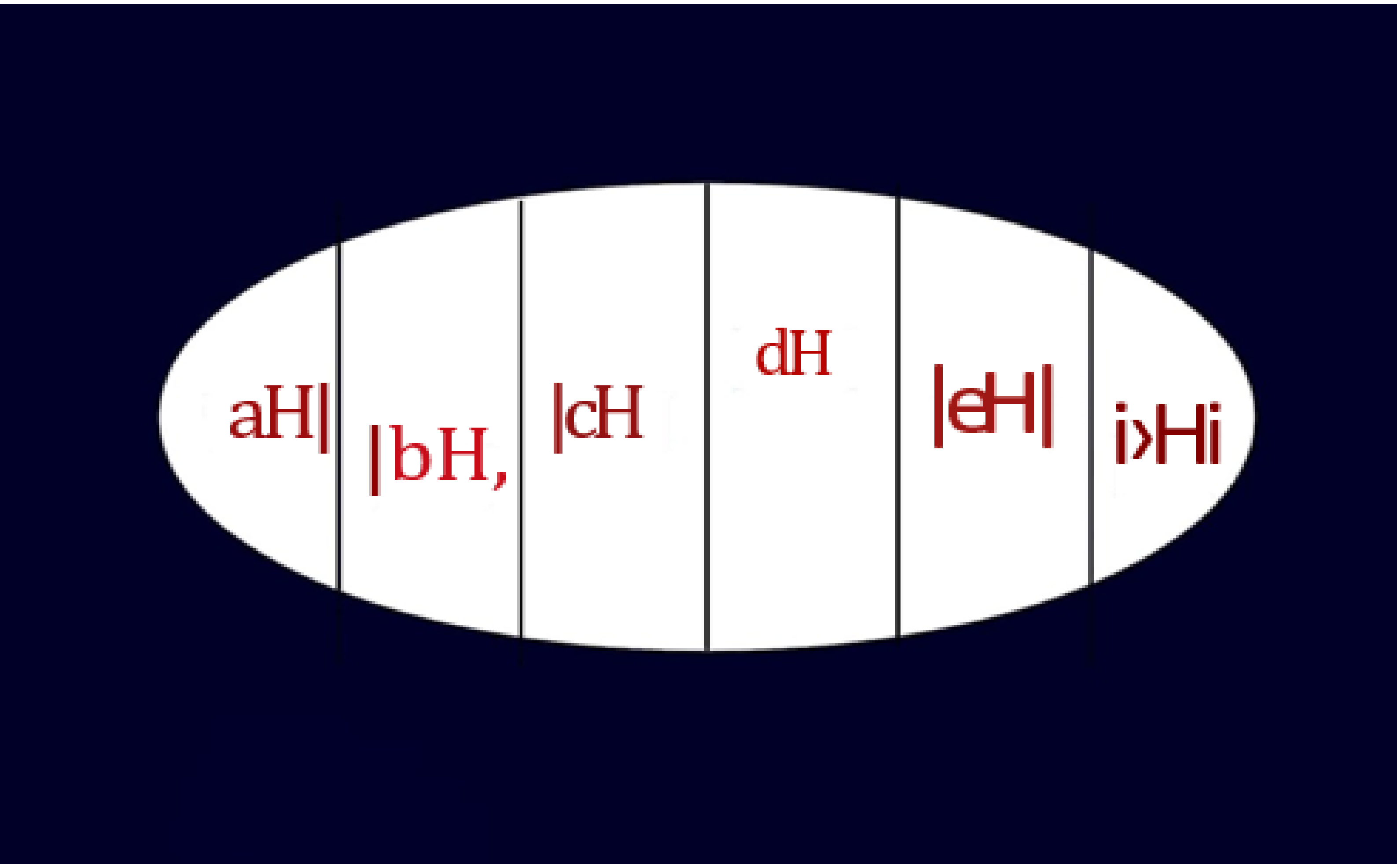
$\Rightarrow aH = bH$

$\underline{f \text{ is onto}}$:

For $Ha \in R_H$. Since $a \in G$, we get $a^{-1} \in G$ and hence $a^{-1}H \in L_H$.

Thus $f(a^{-1}H) = H(a^{-1})^{-1} = Ha$

$\therefore$ f is onto and hence f is bijective.



Recap:

1. $a \in aH$
2. $aH = H \Leftrightarrow a \in H$
3. $a \in Hb \Leftrightarrow Ha = Hb$
4. Either $aH = bH$ or $aH \cap bH = \phi$
5. $aH = bH \Leftrightarrow a^{-1}b \in H$
6. $|aH| = |bH|$
7. $|aH| = |Hb|$